MATHEMATICS

Time Allowed: 3 hours ***Maximum Marks: 300***

Candidates should attempt any FIVE questions.
All questions carry equal marks.

1. (a) Given two linearly independent vectors (1, 0, 1, 0) and (0, -1, 1, 0) of R^4, find a basis of R^4 which includes these two vectors.

   (b) If V is a finite dimensional vector space over R and if f and g are two linear transformations from V to R such that f(v) = 0 implies g(v) = 0, then prove that g = λf for some λ in R.

   (c) Let $T : R^3 \to R^3$ be defined by $T(x_1,x_2,x_3) = (x_2,x_3,-cx_1-bx_2-ax_3)$ where a, b, c are fixed real numbers. Show that T is a linear transformation of R_3 and that

   $$A^3 + aA^2 + bA + cI = 0,$$

   Where A is the matrix of T with respect to standard basis of R^3.

2. (a) If A and B are two matrices of order 2 × 2 such that A is skew Hermitian and AB = B, then show that B =0.

   (b) If T is a complex matrix or order 2×2 such that $\text{tr } T = \text{tr } T^2 = 0$, then show that $T^2 = 0$.

   (c) Prove that a necessary and sufficient condition for a n × n real matrix A to be similar to a diagonal matrix is that the set of characteristic vectors of A includes a set of n linearly independent vectors.

3. (a) Let A be an m×n matrix. Then show that the sum of the rank and nullity of A is n.

   (b) Find all real 2 × 2 matrices A whose characteristic roots are real and which satisfy AA' = I.

   (c) Reduce to diagonal matrix by rational congruent transformation the symmetric matrix

   $$A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 0 & 3 \\ -1 & 3 & 1 \end{pmatrix}$$

4. (a) Find the asymptotes of the curve

   $$(2x-3y+1)^2(x+y)-8x+2y-9=0$$

   and show that they intersect the curve again in three points which lie on a straight line.

   (b) A thin closed rectangular box is to have one edge n times the length of another edge and the volume of the box is given to be v. Prove that the least surface s is given by

   $$ns^3 = 54\,(n+1)^2v^2.$$

   (c) If x + y = 1, Prove that

   $$\frac{d^n}{dx^n}\left(x^n y^n\right) = n!\left[y^n - \binom{n}{1}^2 y^{n-1}x + \binom{n}{2}^2 y^{n-2}x^2 + \ldots + (-1)^n x^n\right]$$

5. (a) Show that

$$\int_0^{\infty}\frac{x^{p-1}}{(1+x)^{p+q}}dx = B(p,q)$$

(b) Show that

$$\iiint \frac{dxdydz}{\sqrt{(1-x^2-y^2-z^2)}} = \frac{\pi^2}{8}$$

Integral being extended over all positive values of x,y,z for which the expression is real.

(c) The ellipse $b^2x^2 + a^2y^2 = a^2b^2$ is divided into two parts by the line $x = \frac{1}{2}a$, and the smaller part is rotated through for right angles about this line. Prove that the volume generated is

$$\pi a^2 b\left\{\frac{3\sqrt{3}}{4} - \frac{\pi}{3}\right\}$$

6. (a) Find the locus of the pole of a chord of the conic $\frac{l}{r} = 1 + e\cos\theta$ which subtends a constant angle 2 α at the focus.

(b) Show that the plane ax+ by+ cz+ d = 0 divides the join of $P_1 \equiv (x_1, y_1, z_1)$, $P_2 \equiv (x_2, y_2, z_2)$ in the ratio $-\frac{ax_1+by_1+cz_1+d}{ax_2+by_2+cz_2+d}$. Hence show that the planes U ≡ ax+by+cz+d= 0 = a'x + b'y+c'z + d' ≡ v, u+λv =0 and u- λv= 0 divide any transversal harmonically.

(c) Prove that a curve x(s) is a generalized helix if and only if it satisfies the identity $x''.x''' \times x^{iv} = 0$.

7. (a) Find the smallest sphere (i.e., the sphere of smallest radius) which touches the lines

$$\frac{x-5}{2} = \frac{y-2}{-1} = \frac{z-5}{-1} \text{ and } \frac{x+4}{-3} = \frac{y+5}{-6} = \frac{z-4}{4}$$

(b) Find the co-ordinates the point of intersection of the generators

$$\frac{x}{a} - \frac{y}{b} - 2\lambda = 0 = \frac{x}{a} - \frac{y}{b} - \frac{z}{\lambda} \text{and} \frac{x}{a} + \frac{y}{b} - 2\mu = 0 = \frac{x}{a} - \frac{y}{b} - \frac{z}{\mu}$$

of the surface $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 2z$.

Hence show that the locus of the points of intersection of perpendicular generators is the curve of intersection of the surface with the plane $2z + (a^2 - b^2) = 0$.

(c) Let P ≡(x, y, z) lie on the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

If the length of the normal chord through P is equal to 4 PG, where G is the intersection of the normal with the z-plane, then show that P lies on the cone

$$\frac{x^2}{a^6}\left(ac^2 - a^2\right) + \frac{y^2}{b^6}\left(ac^2 - b^2\right) + \frac{z^2}{c^4} = 0$$

8. (a) Solve the differential equation:

$$xy - \frac{dy}{dx} = y^3 e^{-x^2}$$

(b) Show that the equation

$$(4x+3y+1)dx+(3x+2y+1)dy=0$$

represents a family of hyperbolas having as asymptotes the lines $x+y=0$, $2x+y+1=0$.

(c) Solve the differential equation: $y = 3px + 4p^2$.

9. (a) Solve the differential equation: $\frac{d^2y}{dx^2}-5\frac{dy}{dx}+6y=e^{4x}\left(x^2+9\right)$

(b) Solve the differential equation: $\frac{d^2y}{dx^2}+2\frac{dy}{dx}+y=x\sin x$

(c) $x^3\frac{d^3y}{dx^3}+2x^2\frac{d^2y}{dx^2}+2y=10\left(x+\frac{1}{x}\right)$

10. (a) If r_1 and r_2 are the vectors joining the fixed points $A(x_1, y_1, z_1)$ $A(x_2, y_2, z_2)$ respectively to a variable point $P(x, y, z)$ then find the values of grad $(r_1 . r_2)$ and $(r_1 \times r_2)$.

(b) Show that $(a \times b) \times c = a \times (b \times c)$ if either $b=0$ (or any other vector is 0) or c is collinear with a or b is orthogonal to a and c (both).

(c) Prove that $\left\{\begin{matrix} i \\ ik \end{matrix}\right\}=\frac{\partial}{\partial x_k}\left(\log\sqrt{g}\right)$

11. (a) A heavy elastic string, whose natural length is $2\pi a$, is places round a smooth cone whose axis is vertical and whose semi-vertical angle is α. If W be the weight and λ the modulus of elasticity of the string, prove that it will be in equilibrium when in b the form of a circle whose radius is $a\left(1+\frac{W}{2\pi\lambda}\cot\alpha\right)$.

(b) Show how to cut out of a uniform cylinder a cone, whose base coincides with that of a cylinder, so that the centre of gravity of the remaining solid may coincide with the vertex of the cone.

(c) One end of an inextensible string is fixed to a point O and to the other end is tied a particle of mass m. The particle is projected from its position of equilibrium vertically below O with a horizontal velocity so as to carry it right round the circle. Prove that the sum of the tensions at the ends of a diameter is constant.

12. (a) Two particles of masses m_1 and m_2 moving in coplanar parabolas round the sun, collide at right angles and coalesce when their common distance from the sun is R. Show that the subsequent path of the combined particles is an ellipse of major axis $(m_1 + m_2)^2 R/2m_1m_2$.

(b) A right circular cone of density ϱ, floats just immersed with its vertex downwards in a vessel containing two liquids of densities σ_1 and σ_2 respectively. Show that the plane of separation of the two liquids cut off from the axis of the cone a fraction

$$\left[\frac{\varrho-\sigma_2}{\sigma_1-\sigma_2}\right]^{1/3}$$ of its length.

(c) A cone floats with its axis horizontal in a liquid of density double its own. Find the pressure on its base and prove that if θ be the inclination to the vertical of the resultant thrust on the curved surface and α the semi vertical angle of the cone,

then $\theta=\tan^{-1}\left[\frac{4}{\pi}\tan\alpha\right]$.

MATHEMATICS

Time Allowed: 3 hours | ***Maximum Marks: 300***

Candidates should attempt any five Questions.
ALL Questions carry equal marks.

PAPER-II

SECTION - A

1. (a) Prove that if a group has only four elements then it must be abelian.

   (b) If H and K are subgroups of a group G then show that HK is a subgroup of G if and only if HK = KH.

   (c) Show that every group of order 15 has a normal subgroup of order 5.

2. (a) Let (R, +, .) be a system satisfying all the axioms for a ring with unity with the possible exception of a + b = b + a. Prove that (R, + , .) is a ring.

   (b) If p is prime then prove that Z_p is a field. Discuss the case when p is not a prime number.

   (c) Let D be a principal domain. Show that every element that its neither zero nor a unit in D is a product of irreducibles.

3. (a) Let X be a metric space and $E \subset X$.. Show that

   (i) interior of E is the largest open set contained in E.

   (ii) boundary of E = (closure of E) $\cap$ (closure of X - E).

   (b) Let (X, d) and (Y, e) be metric spaces with X compact and f: $X \to Y$ be continuous. Show that f is uniformly continuous.

   (c) Show that the function $f(x,y) = 2x^4 - 3x^2y + y^2$ has (0, 0) as the only critical point but the function neither has a minima nor a maxima at (0, 0).

4. (a) Test the convergence of the integral

   $$\int_0^\infty e^{-ax}\frac{\sin x}{x}dx, a \geq 0$$

   (b) Test the series $\sum_{n=1}^{\infty}\frac{x}{\left(n+x^2\right)^2}$ for uniform convergence.

   (c) Let f(x) = x and g(x) = x^2. Does $\int_0^1 fdg$ exists ? If it exists then find its value.

5. (a) Show that the function

   $$f(z) = \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2} z \neq 0$$

   $$f(0) = 0$$

   is continuous and C- R conditions are satisfied at z=0, but f'(z) does not exist at z =0.

(b) Find the Laurent expansion of $\dfrac{z}{(z+1)(z+2)}$ about the singularity z = -2. Specify the region of convergence and the nature of singularity at z = -2.

(c) By using the integral representation of f'(0), prove that

$$\left(\frac{x^n}{\lfloor n}\right)^2 = \frac{1}{2\pi i}\oint_c \frac{x^n e^{xz}}{\lfloor n z^{n+1}} dz$$

where C is any closed contour surrounding, the origin. Hence show that

$$\sum_{n=0}^{\infty}\left(\frac{x^n}{\lfloor n}\right)^2 = \frac{1}{2\pi}\int_0^{2\pi} e^{2x\cos\theta\, d\theta}$$

6. (a) Prove that all roots of $z^7 - 5z^3 + 12 = 0$ lie between the circles |z|=1 and |z|=2.

(b) By integrating round a suitable contour show that

$$\int_0^{\infty} \frac{x\sin mx}{x^4+a^4} dx = \frac{\pi}{4b^2} e^{-mb} \sin mb,$$

where $b = \dfrac{a}{\sqrt{2}}$

(c) Using residue theorem evaluate

$$\int_0^{2\pi} \frac{d\theta}{3-2\cos\theta+\sin\theta}$$

7. (a) (i) Find the differential equation of the set of all right circular cones whose axes whose axes coincide with the Z-axis.

(ii) Form the differential equation by eliminating a, b and c from

Z = a(x+y) + b(x-y) + abt + c.

(b) Solve : $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + w\dfrac{\partial u}{\partial z} = xyz$

(c) Find the integral surface of the linear partial differential equation :

$$x\left(y^2 \mp z\right)\frac{\partial z}{\partial x} - y\left(x^2+z\right)\frac{\partial z}{\partial y} = y\left(x^2-y^2\right)z$$

through the straight line x + y = 0, z = 1.

8. (a) Use Charpit's method to find a complete integral of

$$\left[2x\left(z\frac{\partial z}{\partial y}\right)^2+1\right] = z\frac{\partial z}{\partial x}$$

(b) Find a real function V(x, y),which reduces to zero when y = 0 and satisfies the equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = -4\pi\left(x^2+y^2\right)$$

(c) Apply Jacobi's method to find a complete integral of the equation

$$2\frac{\partial z}{\partial x_1}x_1x_3 + 3\frac{\partial z}{\partial x_2}x_3^2 + \left(\frac{\partial z}{\partial x_2}\right)^2\frac{\partial z}{\partial x_3} = 0$$

SECTION - B

9. (a) Two particles in a plane are connected by a rod of constant length and are constrained to move in such a manner that the velocity of the middle of the rod is in the direction of the rod. Write down the equations of the constraints. Is the system holonomic or non-holonomic ? Give reason for your answer.

(b) Using Lagrannge equations, obtain the differential equations of motion of a free particle in spherical polar coordinates.

(c) A rod of length 2a is suspended by a string of length l attached to one end; if the string and rod revolve about the vertical with uniform angular velocity ω, and their inclinations to the vertical be α and β respectively, show that

$$\omega^2 = \frac{3g \tan \beta}{3l \sin \alpha + 4a \sin \beta}$$

10. (a) A particle of mass m is fixed to a point P of the rim of a uniform circular disc of centre σ, mass m and radius a. The disc is held, with its plane vertical its lowest point in contact with a perfectly rough horizontal table and with OP inclined at 60° to the upward vertical and is then released. If the subsequent motion continues in the same vertical plane, show that, when OP makes and angle θ with the upward vertical

$$a(7 + 4 \cos \theta)\theta^2 = 2g\ (1\text{-}2 \cos \theta).$$

Show also that when OP is first horizontal, the acceleration of α is $\frac{18}{49}g$.

(b) Three equal uniform rods AB, BC, CD each of mass m and length 2a, are at rest in a straight line smoothly jointed at B and C. A blow J is given to the meddle rod at a distance x from its centre σ in a direction perpendicular to it; show that the initial velocity of σ is $\frac{2J}{3m}$, and that the initial angular velocities of the rods are :

$$\frac{5a+9x}{10ma^2}J, \frac{6x}{5ma^2}J, \frac{5a-9x}{10ma^2}J.$$

11. (a) Show that a fluid of constant density can have a velocity $\vec{q}$ given by:

$$\vec{q} = \left[\frac{-2xyz}{\left(x^2+y^2\right)^2} \frac{\left(x^2-y^2\right)z}{\left(x^2+y^2\right)^2} \frac{y}{x^2+y^2}\right]$$

Determine if the fluid motion is irrotational.

(b) Steam is rushing from a boiler through a conical pipe, the diameters of the ends of which are D and d; if V and v be the corresponding velocities of the steam, and if the motion be supposed to be that of divergence from the vertex of the cone, prove that

$$\frac{v}{V} = \left(\frac{D}{d}\right)^2 e^{\frac{v^2-V^2}{2k}}$$

where k is the pressure divided by the density, and supposed constant.

(c) Between two fixed boundaries $\theta = \frac{\pi}{4}$ and $\theta = -\frac{\pi}{4}$ there is a two - dimensional liquid motion due to a source of strength mat the point (r = a, θ = 0), and an equal sink at the point (r = b, θ = 0). Show that the stream function is

$$-m\tan^{-1}\frac{r^4\left(a^4+b^4\right)\cos 4\theta}{r^8-r^4\left(a^4+b^4\right)\cos 4\theta+a^4b^4}.$$

12. (a) Evaluate $\int_1^3 \frac{dx}{x}$ by Simpson's rule with 4 strips.

Determine the error by error by direct integration.

(b) By the fourth - order Runge - Kutta method, tabulate the solution of the differential equation

$$\frac{dy}{dx}=\frac{xy+1}{10y^2+4}, \qquad y(0)=0$$

in [0, .4] with step length .1 correct to give five places of decimals.

(c) Use Regula - Falsi method to show that the real root of $x \log_{10} x - 1.2 = 0$ lies between 3 and 2.740646.

13. (a) A die is tossed. Let X denote twice the number appearing and Y denote 1 or 3, depending on whether an odd or an even number appears. Find the distribution, expectation and variance of X, Y and X+Y.

(b) Let X_1 and X_2 have independent Gamma distributions with parameters α, θ and β, θ respectively. Let $Y_1 = X_1/(X_1+X_2)$ and $Y_2 = X_1 + X_2$. Find p.d.f., $g(y_1, y_2)$ of Y_1 and Y_2. Show that Y_1 has a Beta p.d.f. with parameters α and β.

(c) If x and y are correlated variables and $s_x = s_y$, then find $b_{x,x+y}$ and $b_{x+y,x}$ and hence show that

$$r_{x,x+y}=\sqrt{\frac{1+r}{r}}.$$

14. (a) (i) If X is N(3,16), find $P(4 \le X \le 8)$, $P(0 \le X \le 5)$ and $P(-2 \le X \le 1)$.

(ii) If X is N(25,36), find the constant C such that $P(|X - 25| \le C) = 0.9544$.

$$\left[\text{Given} \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{W^2}{2}} = 0.5987, 0.6915, 0.7734, 0.8944, 0.9772 \quad \text{for } z = 0.25, 0.5, 0.75, 1.25, 2 \text{ respectively.}\right]$$

(b) Fit a second degree parabola to the following data taking x as the independent variable:-

x	y	x	y
1	2	5	10
2	6	6	11
3	7	7	11
4	8	8	10
		9	9

(c) A certain stimulus administered to each of 12 patients resulted in the following increases of blood pressures :-

5, 2, 8, -1, 3, 0, 6, -2, 1, 5, 0, 4.

Can it be calculated that the stimulus will be, in general, accompanied by an increase in blood pressure, given that for 11 degrees of freedom the value $t_{.05}$ is 2.201?

15. (a) Prove that a basic feasible solution to a linear programming problem must corresponds to an extremes point of the set of all, feasible solutions.

(b) Solve the unbalanced assignment problem in minimization where

$$\left[C_{ij}\right]=\begin{bmatrix}12 & 10 & 15 & 22 & 18 & 8\\ 10 & 18 & 25 & 15 & 16 & 12\\ 11 & 10 & 3 & 8 & 5 & 9\\ 6 & 14 & 10 & 13 & 13 & 12\\ 8 & 12 & 11 & 7 & 13 & 10\end{bmatrix}$$

(c) Solve the m×2 game :

$$A=\begin{bmatrix}2 & 3\\ 6 & 7\\ -6 & 10\\ -3 & -2\\ 3 & 2\end{bmatrix}$$

16. (a) Use dynamic programming to find the maximum value of $Z = y_1\ y_2\ y_3$ subject to the constraints: $y_1 + y_2 + y_3 = 5$,

$y_1, y_2, y_3 \geq = 0.$

(b) A bank has two tellers working on savings accounts. The first teller handles withdrawals only. The second teller handles depositors only. It has been found that the service time distributions of both deposits and withdrawals are exponential with a mean service time of 3 minutes per customer. Depositors and withdrawers are found to arrive in a poisson fashion throughout the day with mean arrival rate of 16 and 14 per hour. What would be the effect on the average waiting time for depositors and withdrawers if each teller could handle both withdrawals and deposits ? What would be the effect if this could only be complished by increasing the service time to 3.5 minutes?

(c) A bookbinder processes the manuscripts of five books through the three stages of operation viz., printing, binding and finishing. The time required to perform the printing, binding and finishing operations are given below :-

	Processing Time (in hours)		
Book	**Printing**	**Binding**	**Finishing**
1	50	60	90
2	100	70	110
3	90	30	70
4	70	40	80
5	60	50	10

Determine the order in which books should be processed in order to minimize the total time required to process the books. Find the minimum time.