C.S.E. (MAIN)

MATHEMATICS—2004

(PAPER–I)

Time allowed : 3 hours *Max. Marks : 300*

INSTRUCTIONS

Each question is printed both in Hindi and in English.

Answers must be written in the medium specified in the Admission Certificate issued to you, which must be stated clearly on the cover of the answer-book in the space provided for the purpose. No marks will be given for the answers written in a medium other than that specified in the Admission Certificate.

Candidates should attempt Questions 1 and 5 which are compulsory, and any three of the remaining questions selecting at least one question from each Section.

Assume suitable data if considered necessary and indicate the same clearly.

All questions carry equal marks.

SECTION 'A'

Q. 1. Attempt any *five* of the following :

(a) Let S be space generated by the vectors $\{(0, 2, 6), (3, 1, 6), (4, -2, -2)\}$. What is the dimension of the space S ? Find a basis for S. 12

(b) Show that $f : \mathrm{IR}^3 \to \mathrm{IR}$ is a linear transformation, where $f(x, y, z) = 3x + y - z$. What is the dimension of the kernel ? Find a basis for the kernel. 12

(c) Prove that the function f defined on $[0, 4]$ by $f(x) = [x]$, greatest integer $\leq x$, $x \in [0, 4]$ is integrable on $[0, 4]$ and that $\int_0^4 f(x)dx = 6$.

12

(d) Shaw that : $x - \frac{x^2}{2} < \log(1+x) < x - \frac{x^2}{2(1+x)}, x > 0$. 12

(e) Prove that the locus of the foot of the perpendicular drawn from the vertex on a tangent to the parabola $y^2 = 4\,ax$ is $(x + a)\,y^2 + x^3 = 0$. 12

(f) Find the equations of the tangent planes to the sphere $x^2 + y^2 + z^2 - 4x + 2y - 6z + 5 = 0$, which are parallel to the plane $2x + y - z = 4$. 12

Q. 2. (a) Show that the linear transformation from IR^3 to IR^4

which is represented by the matrix $\begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & -2 \\ 2 & 1 & 1 \\ -1 & 1 & 2 \end{pmatrix}$ is one-to-one.

Find a basis for its image. 15

(b) Verify whether the following system of equations is consistent :

$$x + 3z = 5$$
$$-2x + 5y - z = 0$$
$$-x + 4y + z = 4.$$

15

(c) Find the characteristic polynomial of the matrix $A = \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}$.

Hence find A^{-1} and A^6.

15

(d) Define a positive definite quadratic form. Reduce the quadratic form $x_1^2 + x_3^2 + 2x_1x_2 + 2x_2x_3$ to canonical form. Is this quadratic form positive definite ?

15

Q. 3. (a) Let the roots of the equation in λ.

$(\lambda - x)^3 + (\lambda - y)^3 + (\lambda - z)^3 = 0$

be u, v, w. Prove that

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = -2\frac{(y-z)(z-x)(x-y)}{(u-v)(v-w)(w-u)}.$$

15

(b) Prove that an equation of the form $x^n = \alpha$, where $n e/N$ and $\alpha > 0$ is a real number, has a positive root.

15

(c) Prove that : $\int \frac{x^2+y^2}{p} dx = \frac{\pi ab}{4}[4+(a^2+b^2)(a^{-2}+b^{-2})]$,

when the integral is taken round the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and p is the length of three perpendicular from the centre to the tangent.

15

(d) If the function f is defined by

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0 \quad , & (x,y) = (0,0) \end{cases}$$

then show that f possesses both the partial derivatives at $(0, 0)$ but it is not continuous thereat. 15

Q. 4. (a) Find the locus of the middle points of the chords of the rectangular hyperbola $x^2 - y^2 = a^2$ which touch the parabola $y^2 = 4ax$. 15

(b) Prove that the locus of a line which meets the lines $y = \pm mx$, $z = \pm c$ and the circle $x^2 + y^2 = a^2$, $z = 0$ is $c^2 m^2 (cy - mzx)^2 + c^2 (yz - cmx)^2 = a^2 m^2 (z^2 - c^2)^2$. 15

(c) Prove that the lines of intersection of pairs of tangent planes to $ax^2 + by^2 + cz^2 = 0$ which touch along perpendicular generators lie on the cone $a^2 (b + c) x^2 + b^2 (c + a) y^2 + c^2 (a + b) z^2 = 0$. 15

(d) Tangent planes are drawn to the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ through the point (α, β, γ). prove that the perpendiculars to them through the origin generate the cone $(\alpha x + \beta y + \gamma z)^2 = a^2 x^2 + b^2 y^2 + c^2 z^2$. 15

SECTION 'B'

Q. 5. Attempt any *five* of the following :

(a) Find the solution of the following differential equation

$$\frac{dy}{dx} + y \cos x = \frac{1}{2} \sin 2x.$$ 12

(b) Solve : $y (xy + 2x^2 y^2)\, dx + x (xy - x^2 y^2)\, dy = 0$. 12

(c) A point moving with uniform acceleration describes distances s_1 and s_2 metres in successive intervals of time t_1 and t_2 seconds. Express the acceleration in terms of s_1, s_2, t_1 and t_2. 12

(d) A non uniform string hangs under gravity. Its cross-section at any point is inversely proportional to the tension at that point. Prove that the curve in which the string hangs is an arc of a parabola with its axis vertical. 12

(e) A circular area of radius a is immersed with its plane vertical, and its centre at a depth c. Find the position of its centre of pressure. 12

(f) Show that if $\overline{A}$ and $\overline{B}$ are irrotational, then $\overline{A} \times \overline{B}$ is solenoidal. 12

Q. 6. (a) Solve : $(D^4 - 4D^2 - 5)\, y = e^x (x + \cos x)$. 15

(b) Reduce the equation $(px - y)(py + x) = 2p$, where $p = \dfrac{dy}{dx}$ to Clairaut's equation and hence solve it. 15

(c) Solve : $(x+2)\dfrac{d^2y}{dx^2} - (2x+5)\dfrac{dy}{dx} + 2y = (x+1)e^x$. 15

(d) Solve the following differential equation :

$$(1-x^2)\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} - (1+x^2)y = x.$$ 15

Q. 7. (a) Prove that the velocity required to project a particle from a height h to fall at a horizontal distance a from a point of projection, is at least equal to $\sqrt{g\left[\sqrt{a^2+h^2} - h\right]}$. 15

(b) A car of mass 750 kg is running up a hill of 1 in 30 at a steady speed of 36 km/hr; the friction is equal to the weight of 40 kg. Find the work done in 1 second. 15

(c) A uniform bar AB weights 12 N and rests with one part, AC of length 8 m, on a horizontal table and the remaining part CB projecting over the edge of the table. If the bar is on the point of overbalancing when a weight of 5 N is placed on it at a point 2m from

A and a weight of 7 N is hung from B, find the length of AB. 15

(d) A cone, of given weight and volume, floats with its vertex downwards. Prove that the surface of the cone in contact with the liquid is least when its vertical angle is $2\tan^{-1}\left(\frac{1}{\sqrt{2}}\right)$. 15

Q. 8. (a) Show that the Frenet-Serret formulae can be written in the form

$$\frac{d\overline{T}}{ds} = \overline{\omega} \times \overline{T}, \frac{d\overline{N}}{ds} = \overline{\omega} \times \overline{N} \text{ and } \frac{d\overline{B}}{dx} = \overline{\omega} \times \overline{B},$$

where $\overline{\omega} = \tau\overline{T} + k\overline{B}$. 15

(b) Prove the identity

$$\nabla\left(\overline{A} \cdot \overline{B}\right) = (\overline{B} \cdot \nabla)\overline{A} + (\overline{A} \cdot \nabla)\overline{B} + \overline{B} \times (\nabla \times \overline{A}) + \overline{A} \times (\nabla \times \overline{B}) \cdot$$ 15

(c) Derive the identity

$$\iiint_V (\phi\nabla^2\psi - \psi\nabla^2\phi)dV = \iint_S (\phi\nabla\psi - \psi\nabla\phi).\hat{n}dS,$$

where V is the volume bounded by the closed surface S. 15

(d) Verify Stokes' theorem for

$$\bar{f} = (2x - y)\hat{i} - yz^2\hat{j} - y^2 z\hat{k},$$

where S is the upper half surface of the sphere $x^2 + y^2 + z^2 = 1$ and C is its boundary. 15

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SECTION 'A'

Q. 1. Answer any *five* of the following :

(a) If p is a prime number of the form $4n + 1$, n being a natural number, then show that congruence $x^2 \equiv -1 \bmod p$ is solvable. 12

(b) Let G be a group such that of all $a, b, \in$ G

(i) $ab = ba$ (ii) $(O(a), O(b)) = 1$ then show that $O(ab) = O(a)\, O(b)$. 12

(c) Show that the function $f(x)$ defined as :

$$f(x) = \frac{1}{2^n}, \frac{1}{2^{n+1}} \leq x \leq \frac{1}{2^n}, n = 0,\ 1,\ 2, \ldots\ldots$$

$$f(0) = 0$$

is integrable in [0, 1], although it has an infinite number of points of discontinuity. Show that

$$\int_0^1 f(x)dx = \frac{2}{3}.$$ 12

(d) Show that the function $f(x)$ defined on R by :

$$f(x) = \begin{cases} x \text{ when } x \text{ is irrational} \\ -x \text{ when } x \text{ is rational} \end{cases}$$

is continuous only at $x = 0$. 12

(e) Find the image of the line $y = x$ under the mapping $w = \dfrac{4}{z^2+1}$ and draw the same. Find the points where this transformation ceases to be conformal. 12

(f) Use Simplex method to solve the linear programming problem:

Max. $\quad z = 3x_1 + 2x_2,$

subject to $\quad x_1 + x_2 \leq 4$

$x_1 - x_2 \leq 2$

$x_1, x_2 \geq 0.$ 12

Q. 2. (a) Verify that the set E of the four roots of $x^4 - 1 = 0$ forms a multiplicative group. Also prove that a transformation T, T $(n) = i^n$

is a homomorphism from I_+ (Group of all integers with addition) onto E under multiplication. 10

(b) Prove that if the cancellation law holds for a ring R then $a\,(\neq 0) \in R$ is not a zero divisor and conversely.

(c) The residue class ring $\dfrac{Z}{(m)}$ is a field iff m is a prime integer.

15

(d) Define irreducible element and prime element in an integral domain D with units. Prove that every prime element in D is irreducible and converse of this is not (in general) true. 25

Q. 3. (a) If (x, y, z) be the lengths of perpendiculars drawn from any interior point P of a triangle ABC on the sides BC, CA and AB respectively, then find the minimum value of $x^2 + y^2 + z^2$, the sides of the triangle ABC being a, b, c. 20

(b) Find the volume bounded by the paraboloid $x^2 + y^2 = az$, the cylinder $x^2 + y^2 = 2ay$ and the plane $z = 0$. 20

(c) Let $f(x) \geq g(x)$ for every x in $[a, b]$ and f and g are both bounded and Riemann integrable on $[a, b]$. At a point $c \in [a, b]$, let f and g be continuous and $f(c) > g(c)$ then prove that

$$\int_a^b f(x)dx > \int_a^b g(x)dx$$ and hence show that

$$-\frac{1}{2} < \int_a^b \frac{x^3 \cos 5x}{2+x^2}\,dx < \frac{1}{2}$$ 20

Q. 4. (a) If all zeroes of a polynomial P (z) lie in a half plane then show that zeroes of the derivative P$'(z)$ also lie in the same half plane. 15

(b) Using contour integration evaluate

$$\int_0^{2\pi} \frac{\cos^2 3\theta}{1-2p\cos 2\theta + p^2}\,d\theta,\ 0 < p < 1$$ 15

(c) A travelling salesman has to visit 5 cities. He wishes to start from a particular city, visit each city once and them return to his starting point. Cost of going from one city to another is given below :

$$\begin{array}{c} \\ A \\ B \\ C \\ D \\ E \end{array}\begin{array}{c} \begin{array}{ccccc} A & B & C & D & E \end{array} \\ \begin{bmatrix} \infty & 4 & 10 & 14 & 2 \\ 12 & \infty & 6 & 10 & 4 \\ 16 & 14 & \infty & 8 & 14 \\ 24 & 8 & 12 & \infty & 10 \\ 2 & 6 & 4 & 16 & \infty \end{bmatrix} \end{array}$$

You are required to find the least cost route.

(d) A department has 4 technicians and 4 tasks are to be performed. The technicians differ in efficiency and tasks differ in their intrinsic difficulty. The estimate of time (in hours), each technician would take to perform a task is given below. How should the tasks be allotted, one to a technician, so as to minimize the total work hours ? 15

Technician \ Task	I	II	III	IV
A	8	26	17	11
B	13	28	4	26
C	38	19	18	15
D	19	26	24	10

SECTION 'B'

Q. 5. Attempt any *five* of the following :

(a) Find the integral surface of the following partial differential equation :

$$x(y^2+z)p - y(x^2+z)q = (x^2-y^2)z \qquad 12$$

(b) Find the complete integral of the partial differential equation $(p^2+q^2)x = pz$ and deduce the solution which passes through the curve $x=0$, $z^2=4y$. 12

(c) The velocity of a particle at distance S from a point on its path is given by the following table :

S (meters)	V (m / sec)
0	47
10	58
20	64
30	65
40	61
50	52
60	38

Estimate the time taken to travel the first 60 meters using Simpson's $\frac{1}{3}$ rule. Compare the result with Simpson's $\frac{3}{8}$ rule. 12

(d) (i) If $(AB, CD)_{16} = (x)_2 = (y)_8 = (z)_{10}$ then find x, y and z. 6

(ii) In a 4-bit representation, what is the value of 1111 in signed integer form, unsigned integer form, signed 1's complement form and signed 2's complement form ? 6

(e) A particle of mass m moves under the influence of gravity on the inner surface of the paraboloid of revolution $x^2 + y^2 = az$ which is assumed frictionless. Obtain the equation of motion. Show that it will describe a horizontal circle in the plane $z = h$, provided that it is given an angular velocity whose magnitude is $\omega = \sqrt{2g / a}$. 12

(f) In an incompressible fluid, the vorticity at every point is constant, in magnitude and direction. Do the velocity components satisfy the Laplace equation ? Justify. 12

Q. 6. (a) Solve the partial differential equation

$$\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} - 2\frac{\partial^2 z}{\partial y^2} = (y-1)e^x.$$ 15

(b) A uniform string of length l, held tightly between $x = 0$ and $x = l$ with no initial displacement, is struck at $x = a$, $0 < a < l$, with velocity v_0. Find the displacement of the string at any time $t > 0$. 30

(c) Using Charpit's method, find the complete solution of the partial differential equation $p^2x + q^2y = z$. 15

Q. 7. (a) How many positive and negative roots of the equation $e^x - 5 \sin x = 0$ exist ? Find the smallest positive root correct to 3 decimals, using Newton-Raphson method. 15

(b) Using Gauss-Siedel iterative method, find the solution of the following system :

$$4x - y + 8z = 26$$

$$5x + 2y - z = 6$$

$$x - 10y + 2z = -13$$

upto three iterations. 15

(c) In a certain exam, candidates have to take 2 papers under part A and 2 papers under part B. A candidate has to obtain minimum of 40% in each paper under part A, with an average of 50%, together with a minimum of 35% in each paper under part B, with an average of 40%. For a complete PASS, an overall minimum of 50% is required. Write a BASIC program to declare the result of 100 candidates. 15

(d) Write a BASIC program for solving the differential equation

$$\frac{dy}{dx} = x^2 + y^2, y(0) = 0.1$$

to get $y(x)$, for $0.2 \leq x \leq 5$ at an equal interval of 0.2, by Runge-Kutta fourth order method. 15

Q. 8. (a) Derive the Hamilton equations of motion from the principle of least action and obtain the same for a particle of mass m moving in a force field of potential V.

Write these equations in spherical coordinates (r, θ, ϕ). 30

(b) The space between two infinitely long coaxial cylinders of radii a and b $(b > a)$ respectively is filled by a homogeneous fluid, of density ρ. The inner cylinder is suddenly moved with velocity v perpendicular to this axis, the outer being kept fixed. Show that the resulting impulsive pressure on a length l of inner cylinder is,

$$\pi\rho a^2 l \frac{b^2 + a^2}{b^2 - a^2} v.$$ 30

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