C.S.E. (MAIN)

MATHEMATICS — 2005

PAPER-I

Time allowed : 3 hours *Maximum Marks : 300*

INSTRUCTIONS

Each question is printed both in Hindi and in English.

Answers must be written in the medium specified in the Admission Certificate issued to you, which must be stated clearly on the cover of the answer-book in the space provided for the purpose. No marks will be given for the answers written in a medium other than that specified in the Admission Certificate.

Candidates should attempt Questions 1 and 5 which are compulsory, and any ***three*** *of the remaining questions selecting* ***at least one*** *question from each Section.*

Assume suitable data if considered necessary and indicate the same clearly.

The number of marks carried by each question is indicated at the end of the question.

SECTION 'A'

Q. 1. Attempt any *five* of the following :

(a) Find the values of **k** for which the vectors $(1, 1, 1, 1)$, $(1, 3, -2, \mathbf{k})$, $(2, 2\mathbf{k}-2, -\mathbf{k}-2, 3\mathbf{k}-1)$ and $(3, \mathbf{k}+2, -3, 2\mathbf{k}+1)$ are linearly independent in $\mathbf{R}^4$. 12

(b) Let V be the vector space of polynomials in x of degree $\leq$ n over **R**. Prove that the set $\{1, x, x^2, ..., x^n\}$ is a basis for V. Extend this basis so that it becomes a basis for the set of *all* polynomials in x. 12

(c) Show that the function given below is not continuous at the origin : 12

$$f(x, y) = \begin{cases} 0 & \text{if} \quad xy = 0 \\ 1 & \text{if} \quad xy \neq 0 \end{cases}$$

(d) Let $f : \mathbf{R}^2 \rightarrow \mathbf{R}$ be defined as

$$f(x, y) = \frac{xy}{\sqrt{(x^2 + y^2)}}, (x, y) \neq (0, 0)$$

$$f(0, 0) = 0$$

Prove that f_x and f_y exist at (0, 0), but f is not differentiable at (0, 0). 12

(e) If normals at the points of an ellipse whose eccentric angles are α, β, γ and δ meet in a point, then show that

$$\sin(\beta + \gamma) + \sin(\gamma + \alpha) + \sin(\alpha + \beta) = 0$$ 12

(f) A square ABCD having each diagonal AC and BD of length 2a, is folded along the diagonal AC so that the planes DAC and BAC are at right angle. Find the shortest distance between AB and DC. 12

Q. 2. (a) Let T be a linear transformation on $\mathbf{R}^3$, whose matrix relative to the standard basis of $\mathbf{R}^3$ is

$$\begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & 2 \\ 3 & 3 & 4 \end{bmatrix}$$

Find the matrix of T relative to the basis.

$$\mathcal{B} = \{(1, 1, 1), (1, 1, 0), (0, 1, 1)\}$$

(b) Find the inverse of the matrix given below using elementary row operations only : 15

$$\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$$

(c) If **S** is a skew-Hermitian matrix, then show that $\mathbf{A} = (\mathbf{I} + \mathbf{S})(\mathbf{I} - \mathbf{S})^{-1}$ is a unitary matrix. Also show that every unitary matrix can be expressed in the above form provided -1 is not an eigenvalue of **A**. 15

(d) Reduce the quadratic form

$$6x_1^2 + 3x_2^2 + 3x_3^2 - 4x_1x_2 - 2x_2x_3 + 4x_3x_1$$

to the sum of squares. Also find the corresponding linear transformation, index and signature. 15

Q. 3. (a) If $u = x + y + z$, $uv = y + z$ and $uvw = z$, then find

$$\frac{\partial(x, y, z)}{\partial(u, v, w)}$$ 15

(b) Evaluate

$$\int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}} dx$$

in terms of Beta function. 15

(c) Evaluate $\iiint_V z\,dV$, where V is the volume bounded below by the cone $x^2 + y^2 = z^2$ and above by the sphere $x^2 + y^2 + z^2 = 1$, lying on the positive side of the y-axis. 15

(d) Find the x-coordinate of the centre of gravity of the solid lying inside the cylinder $x^2 + y^2 = 2ax$, between the plane $z = 0$ and the paraboloid $x^2 + y^2 = az$.

Q. 4. (a) A plane is drawn through the line $x + y = 1$, $z = 0$ to make an angle $\sin^{-1}\left(\frac{1}{3}\right)$ with the plane $x + y + z = 5$. Show that two such planes can be drawn. Find their equations and the angle between them. 15

(b) Show that the locus of the centres of spheres of a co-axial system is a straight line. 15

(c) Obtain the equation of a right circular cyclinder on the circle through the points (a, 0, 0), (0, b, 0) and (0, 0, c) as the guiding curve. 15

(d) Reduce the following equation to canonical form and determine which surface is represented by it : 15

$$2x^2 - 7y^2 + 2z^2 - 10\,yz - 8zx - 10\,xy + 6x + 12y - 6z + 2 = 0$$

SECTION 'B'

Q. 5. Attempt any *five* of the following :

(a) Find the orthogonal trajectory of a system of co-axial circles $x^2 + y^2 + 2gx + c = 0$, where g is the parameter. 12

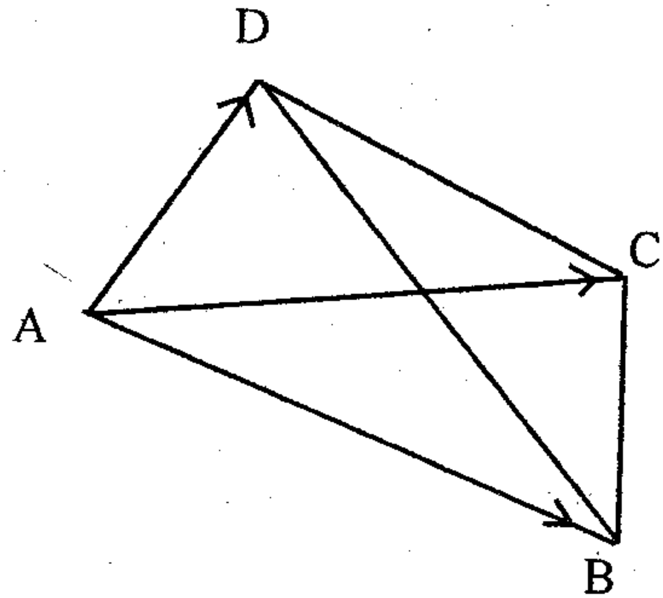
(b) Solve :

$$xy\,\frac{dy}{dx} = \sqrt{(x^2 - y^2 - x^2y^2 - 1)}$$

(c) A body of mass $(m_1 + m_2)$ moving in a straight line is split into two parts of masses m_1 and m_2 by an internal explosion which generates kinetic energy E. If after the explosion, the two parts move in the same line as before, find their relative velocity. 12

(d) If a number of concurrent forces be represented in magnitude and direction by the sides of a closed polygon, taken in order, then show that these forces are in equilibrium. 12

(e) Show that the volume of the tetrahedron ABCD is $\frac{1}{6}$ $(\overrightarrow{AB} \times \overrightarrow{AC}).\overrightarrow{AD}$. Hence, find the volume of the tetrahedron with vertices (2, 2, 2), (2, 0, 0), (0, 2, 0) and (0, 0, 2). 12



(f) Prove that the curl of a vector field is independent of the choice of coordinates. 12

Q. 6. (a) Solve the differential equation : 15

$$[(x+1)^4 D^3 + 2(x+1)^3 D^2 - (x+1)^2 D + (x+1)]\, y = \frac{1}{x+1}$$

(b) Solve the differential equation

$$(x^2 + y^2)(1+p)^2 - 2(x+y)(1+p)(x+yp) + (x+yp)^2 = 0$$

where $p = \dfrac{dy}{dx}$, by reducing it to Clairaut's form by using suitable substitution. 15

(c) Solve the differential equation

$(\sin x - x\cos x)\, y'' - x \sin x y' + y \sin x = 0$ given that $y = \sin x$ is a solution of this equation. 15

(d) Solve the differential equation

$$x^2 y'' - 2xy' + 2y = x \log x,\ x > 0$$

by variation of parameters. 15

Q. 7. (a) A particle is projected along the inner side of a smooth vertical circle of radius a so that its velocity at the lowest point is u. Show that if $2ag < u^2 < 5ag$, the particle will leave the circle before arriving at the highest point and will describe a parabola whose latus rectum is $\dfrac{2(u^2 - 2ga)^3}{27g^3a^2}$ 15

(b) Two particles connected by a fine string are constrained to move in a fine cycloidal tube in a vertical plane. The axis of the cycloid is vertical with vertex upwards. Prove that the tension in the string is constant throughout the motion. 15

(c) Two equal uniform rods AB and AC, of length a each, are freely joined at A, and are placed symmetrically over two smooth pegs on the same horizontal level at a distance c apart ($3c < 2a$). A weight equal to that of a rod, is suspended from the joint A. In the position of equilibrium, find the inclination of either rod with the horizontal by the principle of virtual work. 15

(d) A rectangular lamina of length 2a and breadth 2b is completely immersed in a vertical plane, in a fluid, so that its centre is at a depth h and the side 2a makes an angle α with the horizontal. Find the position of the centre of pressure. 15

Q. 8. (a) The parametric equation of a circular helix is

$$\mathbf{r} = a \cos u\hat{i} + a \sin u\hat{j} + cu\,\hat{k}$$

where c is a constant and u is a parameter. Find the unit tangent vector $\hat{t}$ at the point u and the arc length measured from $u = 0$.

Also find $\dfrac{d\hat{t}}{ds}$, where s is the arc length. 15

(b) Show that

$$\text{curl}\left(\mathbf{k} \times \text{grad}\frac{1}{r}\right) + \text{grad}\left(\mathbf{k}.\text{grad}\frac{1}{r}\right) = 0$$

where r is the distance from the origin and **k** is the unit vector in the direction OZ. 15

(c) Find the curvature and the torsion of the space curve

$$x = a(3u - u^3)$$
$$y = 3au^2$$
$$z = a(3u + u^3)$$ 15

(d) Evaluate

$$\oiint_S (x^3\,dy\,dz + x^2 y\,dz\,dx + x^2 z\,dx\,dy)$$

by Gauss divergence theorem, where S is the surface of the cylinder $x^2 + y^2 = a^2$ bounded by $z = 0$ and $z = b$. 15

SECTION 'A'

Q. 1. Answer any *five* of the following :

(a) If M and N are normal subgroups of a group G such that $M \cap N = \{e\}$, show that every element of M commutes with every element of N. 12

(b) Show that $(1 + i)$ is a prime element in the ring R of Gaussian integers. 12

(c) If u, v, w are the roots of the equation in λ and

$$\frac{x}{a+\lambda}+\frac{y}{b+\lambda}+\frac{z}{c+\lambda}=1, \text{ evaluate } \frac{\partial(x, y, z)}{\partial(u, v, w)}.$$ 12

(d) Evaluate $\iiint \ln(x+y+z)\, dx\, dy\, dz$.

The integral being extended over all positive values of x, y, z such that $x + y + z \leq 1$. 12

(e) If $f(z) = u + iv$ is an analytic function of the complex variable z and $u - v = e^x(\cos y - \sin y)$, determine $f(z)$ in terms of z. 12

(f) Put the following program in standard form :

$$\begin{aligned} \text{Minimize } z &= 25x_1 + 30x_2 \\ \text{subject to } 4x_1 + 7x_2 &\geq 1 \\ 8x_1 + 5x_2 &\geq 3 \\ 6x_1 + 9x_2 &\geq -2 \end{aligned}$$

and hence obtain an initial feasible solution. 12

Q. 2. (a) (i) Let H and K be two subgroups of a finite group G such that $|H| > \sqrt{|G|}$ and $|K| > \sqrt{|G|}$. Prove that $H \cap K \neq \{e\}$. 15

(ii) If $f : G \to G'$ is an isomorphism, prove that the order of $a \in G$ is equal to the order of $f(a)$. 15

(b) Prove that any polynomial ring F [x] over a field F is a

30

Q. 3. (a) If f' and g' exist for every $x \in [a, b]$ and if $g'(x)$ does not vanish anywhere in (a, b), show that there exists c in (a, b) such that

$$\frac{f(c)-f(a)}{g(b)-g(c)}=\frac{f'(c)}{g'(c)}$$ 30

(b) Show that $\int_0^\infty e^{-t}\, t^{n-1}\, dt$ is an improper integral which converges for $n > 0$. 30

Q. 4. (a) Expand $f(z)=\frac{1}{(z+1)(z+3)}$ in Laurent's series which is valid for

(i) $1<|z|<3$

(ii) $|z|>3$ and

(iii) $|z|<1$. 30

(b) Use simplex method to solve the following : 30

Maximize $z=5x_1+2x_2$

subject to $6x_1+x_2 \geq 6$

$4x_1+3x_2 \geq 12$

$x_1+2x_2 \geq 4$

and $x_1, x_2 \geq 0$.

SECTION 'B'

Q. 5. Answer any *five* of the following :

(a) Formulate partial differential equation for surfaces whose tangent planes form a tetrahedron of constant volume with the co-ordinate planes. 12

(b) Find the particular integral of

$$x(y-z)\,p+y(z-x)\,q=z(x-y)$$

which represents a surface passing through x = y = z. 12

(c) Use appropriate quadrature formulae out of the Trapezoidal and Simpson's rules to numerically integrate $\int_0^1 \frac{dx}{1+x^2}$ with $h = 0.2$

Hence obtain an approximate value of π. Justify the use of a particular quadrature formula. 12

(d) Find the hexadecimal equivalent of $(41819)_{10}$ and decimal equivalent of $(111011.10)_2$.

(e) A rectangular plate swings in a vertical plane about one of its corners. If its period is one second, find the length of its diagonal. 12

(f) Prove that the necessary and sufficient condition for vortex lines and stream lines to be at right angles to each other is that

$$u = \mu \frac{\partial \varphi}{\partial x}, \; v = \mu \frac{\partial \varphi}{\partial y}, \; w = \mu \frac{\partial \varphi}{\partial z}$$

where μ and φ are functions of x, y, z and t. 12

Q. 6. (a) The ends A and B of a rod 20 cm long have the temperature at 30° C and at 80° C until steady state prevails. The temperature of the ends are changed to 40° C and 60° C respectively. Find the temperature distribution in the rod at time t. 30

(b) Obtain the general solution of

$$(D - 3D' - 2)^2 z = 2\, e^{2x} \sin (y + 3x).$$

where $D = \frac{\partial}{\partial x}$ and $D' = \frac{\partial}{\partial y}$ 30

Q. 7. (a) Find the unique polynomial P (x) of degree 2 or less such that P (1) = 1, P (3) = 27, P (4) = 64. Using the Lagrange interpolation formula and the Newton's divided difference formula, evaluate P (1.5). 30

(b) Draw a flow chart and also write a program in BASIC to find one real root of the non linear equation $x = \varphi(x)$ by the fixed point iteration method. Illustrate it to find one real root, correct upto four places of decimals, of $x^3 - 2x - 5 = 0$. 30

Q. 8. (a) A plank of mass M, is initially at rest along a line of greatest slope of a smooth plane inclined at an angle α to the horizon, and a man of mass M′ starting from the upper end walks down the plank so that it does not move. Show that he gets to the other end in time

$$\sqrt{\frac{2M'a}{(M+M')\, g \sin\alpha}}$$

where a is the length of the plank. 30

(b) State the conditions under which Euler's equations of motion can be integrated. Show that

$$-\frac{\partial\varphi}{\partial t} + \frac{1}{2}q^2 + V + \int \frac{dp}{\rho} = F(t)$$

where the symbols have their usual meaning. 30