Indian Forest Service Examination-2012

D-GT-M-NUA

MATHEMATICS

Paper I

Time Allowed : Three Hours	*Maximum Marks : 200*

INSTRUCTIONS

Candidates should attempt Question Nos. 1 and 5 which are compulsory and any THREE of the remaining questions, selecting at least ONE question from each Section.

All questions carry equal marks.

Marks allotted to parts of a question are indicated against each.

Answers must be written in ENGLISH only.

Assume suitable data, if considered necessary, and indicate the same clearly.

Unless indicated otherwise, symbols & notations carry their usual meaning.

Important Note : All parts/sub-parts of a question must be attempted contiguously. That is, candidates must finish attempting all the parts/sub-parts of each question they are answering in the answer-book before moving on to the next question.

Pages left blank must be clearly struck out. Answers that follow any pages left blank may not be given credit.

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Section – A

1. (a) Let $V = \mathbb{R}^3$ and $\alpha_1 = (1, 1, 2)$, $\alpha_2 = (0, 1, 3)$, $\alpha_3 = (2, 4, 5)$ and $\alpha_4 = (-1, 0, -1)$ be the elements of V. Find a basis for the intersection of the subspace spanned by $\{\alpha_1, \alpha_2\}$ and $\{\alpha_3, \alpha_4\}$. 8

(b) Show that the set of all functions which satisfy the differential equation

$\frac{d^2 f}{dx^2} + 3\frac{df}{dx} = 0$ is a vector space. 8

(c) If the three thermodynamic variables P, V, T are connected by a relation, $f(P, V, T) = 0$ show that, $\left(\frac{\partial P}{\partial T}\right)_V \left(\frac{\partial T}{\partial V}\right)_P \left(\frac{\partial V}{\partial P}\right)_T = -1$. 8

(d) If $u = Ae^{-gx}\sin(nt - gx)$, where A, g, n are positive constants, satisfies the heat conduction equation, $\frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial x^2}$ then show that

$g = \sqrt{\left(\frac{n}{2\mu}\right)}$. 8

(e) Find the equations to the lines in which the plane $2x + y - z = 0$ cuts the cone

$$4x^2 - y^2 + 3z^2 = 0.$$ 8

2\. (a) Let $f: \mathbb{R} \to \mathbb{R}^3$ be a linear transformation defined by $f(a, b, c) = (a, a+b, 0)$.

Find the matrices A and B respectively of the linear transformation f with respect to the standard basis (e_1, e_2, e_3) and the basis (e_1', e_2', e_3') where $e_1' = (1, 1, 0)$, $e_2' = (0, 1, 1)$, $e_3' = (1, 1, 1)$.

Also, show that there exists an invertible matrix P such that

$$B = P^{-1} A P$$ 10

(b) Verify Cayley–Hamilton theorem for the matrix $A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$ and find its inverse. Also express $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I$ as a linear polynomial in A. 10

(c) Find the equations of the tangent plane to the ellipsoid $2x^2 + 6y^2 + 3z^2 = 27$ which passes through the line $x - y - z = 0 = x - y + 2z - 9$. 10

(d) Show that there are three real values of λ for which the equations :
$(a - \lambda)x + by + cz = 0$, $bx + (c - \lambda)y + az = 0$, $cx + ay + (b - \lambda)z = 0$ are simultaneously true and that the product of these values of λ is

$$D = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$ 10

3. (a) Find the matrix representation of linear transformation T on $V_3(\mathbb{R})$ defined as $T(a, b, c) = (2b + c, a - 4b, 3a)$ corresponding to the basis $B = \{(1, 1, 1), (1, 1, 0), (1, 0, 0)\}$. 10

(b) Find the dimensions of the rectangular box, open at the top, of maximum capacity whose surface is 432 sq. cm. 10

(c) If $2C$ is the shortest distance between the lines

$$\frac{x}{l} - \frac{z}{n} = 1,\ y = 0$$

and $\frac{y}{m} + \frac{z}{n} = 1,\ x = 0$

then show that

$$\frac{1}{l^2} + \frac{1}{m^2} + \frac{1}{n^2} = \frac{1}{c^2}.$$ 10

(d) Show that the function defined as

$$f(x) = \begin{cases} \dfrac{\sin 2x}{x} & \text{when } x \neq 0 \\ 1 & \text{when } x = 0 \end{cases}$$

has removable discontinuity at the origin. 10

4. (a) Find by triple Integration the volume cut off from the cylinder $x^2 + y^2 = ax$ by the planes $z = mx$ and $z = nx$. 10

(b) Show that all the spheres, that can be drawn through the origin and each set of points where planes parallel to the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=0$ cut the co-ordinate axes, form a system of spheres which are cut orthogonally by the sphere

$$x^2+y^2+2fx+2gy+2hz=0$$

if $af+bg+ch=0$. 10

(c) A plane makes equal intercepts on the positive parts of the axes and touches the ellipsoid $x^2+4y^2+9z^2=36$. Find its equation. 10

(d) Evaluate the following in terms of Gamma function :

$$\int_0^a \sqrt{\left(\frac{x^3}{a^3-x^3}\right)}\,dx.$$ 10

Section – B

5. (a) Solve $\frac{dy}{dx}-\frac{\tan y}{1+x}=(1+x)e^x\sec y$. 8

(b) Solve and find the singular solution of $x^3p^2+x^2py+a^3=0$. 8

(c) A particle is projected vertically upwards from the earth's surface with a velocity just sufficient to carry it to infinity. Prove that the time it takes to reach a height h is

$$\frac{1}{3}\sqrt{\left(\frac{2a}{g}\right)}\left[\left(1+\frac{h}{a}\right)^{3/2}-1\right].$$ 8

(d) A triangle ABC is immersed in a liquid with the vertex C in the surface and the sides AC, BC equally inclined to the surface. Show that the vertical C divides the triangle into two others, the fluid pressures on which are as $b^3 + 3ab^2 : a^3 + 3a^2b$ where a and b are the sides BC & AC respectively. 8

(e) If $u = x + y + z$, $v = x^2 + y^2 + z^2$, $w = yz + zx + xy$, prove that grad u, grad v and grad w are coplanar. 8

6. (a) Solve :

$$x^2 y \frac{d^2 y}{dx^2} + \left(x \frac{dy}{dx} - y \right)^2 = 0.$$ 10

(b) Find the value of $\iint_S \left(\vec{\nabla} \times \vec{F} \right) \cdot \vec{ds}$

taken over the upper portion of the surface $x^2 + y^2 - 2ax + az = 0$ and the bounding curve lies in the plane $z = 0$, when

$$\vec{F} = \left(y^2 + z^2 - x\right) \vec{i} + \left(z^2 + x^2 - y^2\right) \vec{j} + \left(x^2 + y^2 - z^2\right) \vec{k}.$$

10

(c) A particle is projected with a velocity u and strikes at right angle on a plane through the plane of projection inclined at an angle β to the horizon. Show that the time of flight is

$$\frac{2u}{g\sqrt{\left(1+3\sin^2\beta\right)}},$$

range on the plane is $\dfrac{2u^2}{g}\cdot\dfrac{\sin\beta}{1+3\sin^2\beta}$

and the vertical height of the point struck is

$\dfrac{2u^2\sin^2\beta}{g\left(1+3\sin^2\beta\right)}$ above the point of projection.

10

(d) Solve $\dfrac{d^4y}{dx^4}+2\dfrac{d^2y}{dx^2}+y=x^2\cos x$. 10

7. (a) A particle is moving with central acceleration $\mu\left[r^5-c^4r\right]$ being projected from an apse at a distance c with velocity $\sqrt{\left(\dfrac{2\mu}{3}\right)}c^3$, show that its path is a curve, $x^4+y^4=c^4$. 13

(b) A thin equilateral rectangular plate of uniform thickness and density rests with one end of its base on a rough horizontal plane and the other against a small vertical wall. Show that the least angle, its base can make with the horizontal plane is given by

$$\cot\theta=2\mu+\frac{1}{\sqrt{3}}$$

μ, being the coefficient of friction. 14

(c) A semicircular area of radius a is immersed vertically with its diameter horizontal at a depth b. If the circumference be below the centre, prove that the depth of centre of pressure is

$$\frac{1}{4}\frac{3\pi\left(a^2+4b^2\right)+32ab}{4a+3\pi b}.$$ 13

8. (a) Solve $x = y\frac{dy}{dx} - \left(\frac{dy}{dx}\right)^2$. 10

(b) Find the value of the line integral over a circular path given by $x^2 + y^2 = a^2$, $z = 0$, where the vector field,

$$\vec{F} = (\sin y)\,\vec{i} + x(1+\cos y)\,\vec{j}.$$ 10

(c) A heavy elastic string, whose natural length is $2\pi a$, is placed round a smooth cone whose axis is vertical and whose semi vertical angle is α. If W be the weight and λ the modulus of elasticity of the string, prove that it will be in equilibrium when in the form of a circle whose radius is

$$a\left(1+\frac{W}{2\pi\lambda}\cot\alpha\right).$$ 10

(d) Solve $x^2\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + y = (1-x)^{-2}$. 10

D–GT–M–NUB

MATHEMATICS

Paper—II

Time Allowed : Three Hours | *Maximum Marks : 200*

INSTRUCTIONS

Candidates should attempt Question Nos. 1 and 5 which are compulsory, and any ***THREE*** *of the remaining questions, selecting at least* ***ONE*** *question from each Section.*

All questions carry equal marks.

The number of marks carried by each part of a question is indicated against each.

Answers must be written in ***ENGLISH*** *only.*

Assume suitable data, if considered necessary, and indicate the same clearly.

Symbols and notations have their usual meanings, unless indicated otherwise.

Important Note :—

All parts/sub-parts of a question must be attempted contiguously. That is, candidates must complete attempting all parts/sub-parts of a question being answered in the answer book before moving on to the next question.

Pages left blank, if any, in the answer-book(s) must be clearly struck out. Answers that follow pages left blank may not be given credit.

SECTION—A

1. Answer the following :

(a) Show that the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & , \quad x \text{ is irrational} \\ -1 & , \quad x \text{ is rational} \end{cases}$$

is discontinuous at every point in $\mathbb{R}$. 10

(b) Show that every field is without zero divisor. 10

(c) Evaluate the integral

$$\int_{2-i}^{4+i} (x + y^2 - ixy)\, dz$$

along the line segment AB joining the points A(2, −1) and B(4, 1). 10

(d) Show that the functions :

$$u = x^2 + y^2 + z^2$$
$$v = x + y + z$$
$$w = yz + zx + xy$$

are not independent of one another. 10

2. (a) Show that in a symmetric group S_3, there are four elements σ satisfying σ^2 = Identity and three elements satisfying σ^3 = Identity. 13

(b) If

$$u = x^2 \tan^{-1}\left(\frac{y}{x}\right) - y^2 \tan^{-1}\left(\frac{x}{y}\right),$$

show that

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = 2u.$$ 13

(c) Solve the following problem by Simplex Method. How does the optimal table indicate that the optimal solution obtained is not unique ?

Maximize $z = 8x_1 + 7x_2 - 2x_3$

subject to the constraints

$$x_1 + 2x_2 + 2x_3 \le 12$$
$$2x_1 + x_2 - 2x_3 \le 12$$
$$x_1, x_2, x_3 \ge 0$$ 14

3. (a) Find the volume of the solid bounded above by the parabolic cylinder $z = 4 - y^2$ and bounded below by the elliptic paraboloid $z = x^2 + 3y^2$. 13

(b) Show that the function

$$u(x, y) = e^{-x}(x \cos y + y \sin y)$$

is harmonic. Find its conjugate harmonic function $v(x, y)$ and the corresponding analytic function $f(z)$. 13

(c) If R is an integral domain, show that the polynomial ring $R[x]$ is also an integral domain. 14

4. (a) Using the Residue Theorem, evaluate the integral

$$\int_C \frac{e^z - 1}{z(z-1)(z+i)^2} \, dz,$$

where C is the circle $|z| = 2$. 13

(b) Examine the series

$$\sum_{n=1}^{\infty} u_n(x) = \sum_{n=1}^{\infty} \left[\frac{nx}{1+n^2x^2} - \frac{(n-1)x}{1+(n-1)^2x^2} \right]$$

for uniform convergence. Also, with the help of this example, show that the condition of uniform convergence of $\sum_{n=1}^{\infty} u_n(x)$ is sufficient but not necessary for the sum $S(x)$ of the series to be continuous. 13

(c) Find the initial basic feasible solution of the following minimum cost transportation problem by Least Cost (Matrix Minima) Method and using it find the *optimal* transportation cost :—

Destinations

		D_1	D_2	D_3	D_4	Supply
	S_1	5	11	12	13	10
Sources	S_2	8	12	7	8	30
	S_3	12	7	15	6	35
Requirement		15	15	20	25	

14

SECTION—B

5. Answer the following :

(a) Using Lagrange's interpolation formula, show that $32f(1) = -3f(-4) + 10f(-2) + 30f(2) - 5f(4)$.

10

(b) Solve

$$(D^3D'^2 + D^2D'^3)z = 0,$$

where D stands for $\frac{\partial}{\partial x}$ and D' stands for $\frac{\partial}{\partial y}$.

10

(Contd.)

(c) Write a computer program to implement trapezoidal rule to evaluate

$$\int_0^{10}\left(1-e^{-\frac{x}{2}}\right)dx. \qquad 10$$

(d) Prove that the vorticity vector $\vec{\Omega}$ of an incompressible viscous fluid moving in the absence of an external force satisfies the differential equation

$$\frac{D\vec{\Omega}}{Dt} = \left(\vec{\Omega}\cdot\nabla\right)\vec{q} + \nu\ \nabla^2\vec{\Omega},$$

where ν is kinematic viscosity. 10

6. (a) Using Method of Separation of Variables, solve Laplace Equation in three dimensions. 13

(b) Derive the differential equation of motion for a spherical pendulum. 13

(c) A river is 80 meters wide. The depth d (in meters) of the river at a distance x from one bank of the river is given by the following table :

x	0	10	20	30	40	50	60	70	80
d	0	4	7	9	12	15	14	8	3

Find approximately the area of cross-section of the river. 14

7. (a) Show that

$$u = \frac{A(x^2 - y^2)}{(x^2 + y^2)^2}, \; v = \frac{2Axy}{(x^2 + y^2)^2}, \quad w = 0$$

are components of a possible velocity vector for inviscid incompressible fluid flow. Determine the pressure associated with this velocity field. 13

(b) Solve the following system of equations using Gauss-Seidel Method :

$$28x + 4y - z = 32$$

$$2x + 17y + 4z = 35$$

$$x + 3y + 10z = 24$$

correct to three decimal places. 13

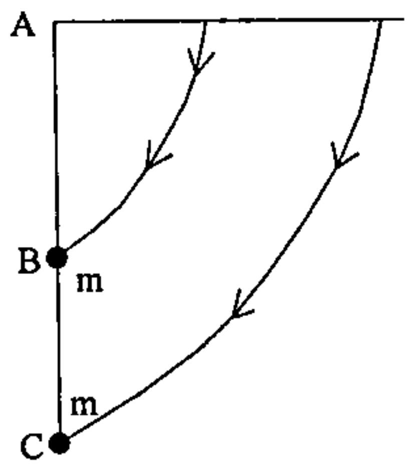
(c) Draw a flow chart for interpolation using Newton's forward difference formula. 14

8. (a) Solve

$$(x^2 - yz)p + (y^2 - zx)q = z^2 - xy$$

using Lagrange's Method. 13

(b) A weightless rod ABC of length 2a is movable about the end A which is fixed and carries two particles of mass m each one attached to the mid-point B of the rod and the other attached to the end C of the rod. If the rod is held in the horizontal position and released from rest and allowed to move, show that the angular velocity of the rod when it is vertical is $\sqrt{\frac{6g}{5a}}$. 13



(c) Using Euler's Modified Method, obtain the solution of

$$\frac{dy}{dx} = x + \left|\sqrt{y}\right|, \quad y(0) = 1$$

for the range $0 \leq x \leq 0.6$ and step size 0.2.

14