I.F.S. EXAM-2016

C-MNS-U-MTS

MATHEMATICS

Paper – I

Time Allowed : Three Hours

Maximum Marks : 200

Question Paper Specific Instructions

Please read each of the following instructions carefully before attempting questions :

*There are **EIGHT** questions in all, out of which **FIVE** are to be attempted.*

*Questions no. **1** and **5** are compulsory. Out of the remaining **SIX** questions, **THREE** are to be attempted selecting at least **ONE** question from each of the two Sections A and B.*

Attempts of questions shall be counted in sequential order. Unless struck off, attempt of a question shall be counted even if attempted partly. Any page or portion of the page left blank in the Question-cum-Answer Booklet must be clearly struck off.

All questions carry equal marks. The number of marks carried by a question/part is indicated against it.

*Answers must be written in **ENGLISH** only.*

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Assume suitable data, if necessary, and indicate the same clearly.

SECTION A

Q1. (a) Let $T : \mathbb{R}^3 \to \mathbb{R}^4$ be given by

$T(x, y, z) = (2x - y,\ 2x + z,\ x + 2z,\ x + y + z).$

Find the matrix of T with respect to standard basis of $\mathbb{R}^3$ and $\mathbb{R}^4$ (i.e., (1, 0, 0), (0, 1, 0), etc.). Examine if T is a linear map. *8*

(b) Show that $\dfrac{x}{(1+x)} < \log(1+x) < x$ for $x > 0$. *8*

(c) Examine if the function $f(x, y) = \dfrac{xy}{x^2 + y^2}$, $(x, y) \neq (0, 0)$ and $f(0, 0) = 0$ is continuous at (0, 0). Find $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ at points other than origin. *8*

(d) If the point (2, 3) is the mid-point of a chord of the parabola $y^2 = 4x$, then obtain the equation of the chord. *8*

(e) For the matrix $A = \begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$, obtain the eigen value and get the value of $A^4 + 3A^3 - 9A^2$. *8*

Q2. (a) After changing the order of integration of $\int_0^{\infty}\int_0^{\infty} e^{-xy} \sin nx \; dx\, dy$, show that $\int_0^{\infty} \frac{\sin nx}{x} dx = \frac{\pi}{2}$. *10*

(b) A perpendicular is drawn from the centre of ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ to any tangent. Prove that the locus of the foot of the perpendicular is given by $(x^2 + y^2)^2 = a^2x^2 + b^2y^2$. *10*

(c) Using mean value theorem, find a point on the curve $y = \sqrt{x-2}$, defined on [2, 3], where the tangent is parallel to the chord joining the end points of the curve. *10*

(d) Let T be a linear map such that $T : V_3 \to V_2$ defined by $T(e_1) = 2f_1 - f_2$, $T(e_2) = f_1 + 2f_2$, $T(e_3) = 0f_1 + 0f_2$, where e_1, e_2, e_3 and f_1, f_2 are standard basis in V_3 and V_2. Find the matrix of T relative to these basis.

Further take two other basis $B_1[(1, 1, 0)\ (1, 0, 1)\ (0, 1, 1)]$ and $B_2[(1, 1)\ (1, -1)]$. Obtain the matrix T_1 relative to B_1 and B_2. *10*

Q3. (a) For the matrix $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$, find two non-singular matrices P and Q such that $PAQ = I$. Hence find A^{-1}. *10*

(b) Using Lagrange's method of multipliers, find the point on the plane $2x + 3y + 4z = 5$ which is closest to the point $(1, 0, 0)$. *10*

(c) Obtain the area between the curve $r = 3(\sec\theta + \cos\theta)$ and its asymptote $x = 3$. *10*

(d) Obtain the equation of the sphere on which the intersection of the plane $5x - 2y + 4z + 7 = 0$ with the sphere which has $(0, 1, 0)$ and $(3, -5, 2)$ as the end points of its diameter is a great circle. *10*

Q4. (a) Examine whether the real quadratic form $4x^2 - y^2 + 2z^2 + 2xy - 2yz - 4xz$ is a positive definite or not. Reduce it to its diagonal form and determine its signature. *10*

(b) Show that the integral $\int_0^\infty e^{-x} x^{\alpha-1}\, dx$, $\alpha > 0$ exists, by separately taking the cases for $\alpha \geq 1$ and $0 < \alpha < 1$. *10*

(c) Prove that $\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}} \Gamma(z)\, \Gamma\left(z + \frac{1}{2}\right)$. *10*

(d) A plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ cuts the coordinate plane at A, B, C. Find the equation of the cone with vertex at origin and guiding curve as the circle passing through A, B, C. *10*

SECTION B

Q5. (a) Obtain the curve which passes through (1, 2) and has a slope = $\frac{-2xy}{x^2+1}$.

Obtain one asymptote to the curve. *8*

(b) Solve the dE to get the particular integral of $\frac{d^4y}{dx^4} + 2\frac{d^2y}{dx^2} + y = x^2 \cos x$. *8*

(c) A weight W is hanging with the help of two strings of length l and $2l$ in such a way that the other ends A and B of those strings lie on a horizontal line at a distance $2l$. Obtain the tension in the two strings. *8*

(d) From a point in a smooth horizontal plane, a particle is projected with velocity u at angle α to the horizontal from the foot of a plane, inclined at an angle β with respect to the horizon. Show that it will strike the plane at right angles, if $\cot \beta = 2 \tan (\alpha - \beta)$. *8*

(e) If E be the solid bounded by the xy plane and the paraboloid $z = 4 - x^2 - y^2$, then evaluate $\iint_S \bar{F} . dS$ where S is the surface bounding the volume E and $\bar{F} = (zx \sin yz + x^3)\,\hat{i} + \cos yz\,\hat{j} + (3zy^2 - e^{\lambda^2 + y^2})\,\hat{k}$. *8*

Q6. (a) A stone is thrown vertically with the velocity which would just carry it to a height of 40 m. Two seconds later another stone is projected vertically from the same place with the same velocity. When and where will they meet ? *10*

(b) Using the method of variation of parameters, solve

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = x^2 e^x .$$ *10*

(c) Water is flowing through a pipe of 80 mm diameter under a gauge pressure of 60 kPa, with a mean velocity of 2 m/s. Find the total head, if the pipe is 7 m above the datum line. *10*

(d) Evaluate $\iint_S (\nabla \times \bar{f}) . \hat{n}\, dS$ for $\bar{f} = (2x - y)\,\hat{i} - yz^2\,\hat{j} - y^2z\,\hat{k}$ where S is the upper half surface of the sphere $x^2 + y^2 + z^2 = 1$ bounded by its projection on the xy plane. *10*

Q7. (a) State Stokes' theorem. Verify the Stokes' theorem for the function $\overline{f} = x\hat{i} + z\hat{j} + 2y\hat{k}$, where c is the curve obtained by the intersection of the plane $z = x$ and the cylinder $x^2 + y^2 = 1$ and S is the surface inside the intersected one. *15*

(b) A uniform rod of weight W is resting against an equally rough horizon and a wall, at an angle α with the wall. At this condition, a horizontal force P is stopping them from sliding, implemented at the mid-point of the rod. Prove that $P = W \tan(\alpha - 2\lambda)$, where λ is the angle of friction. Is there any condition on λ and α ? *15*

(c) Obtain the singular solution of the differential equation

$$y^2 - 2pxy + p^2(x^2 - 1) = m^2, \ p = \frac{dy}{dx}.$$ *10*

Q8. (a) A body immersed in a liquid is balanced by a weight P to which it is attached by a thread passing over a fixed pulley and when half immersed, is balanced in the same manner by weight 2P. Prove that the density of the body and the liquid are in the ratio 3 : 2. *10*

(b) Solve the differential equation

$$\frac{dy}{dx} - y = y^2 (\sin x + \cos x).$$ *10*

(c) Prove that $\overline{a} \times (\overline{b} \times \overline{c}) = (\overline{a} \times \overline{b}) \times \overline{c}$, if and only if either $\overline{b} = \overline{0}$ or $\overline{c}$ is collinear with $\overline{a}$ or $\overline{b}$ is perpendicular to both $\overline{a}$ and $\overline{c}$. *10*

(d) A particle is acted on a force parallel to the axis of y whose acceleration is λy, initially projected with a velocity $a\sqrt{\lambda}$ parallel to x-axis at the point where $y = a$. Prove that it will describe a catenary. *10*

I.F.S. EXAM-2016

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MATHEMATICS

Paper II

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QUESTION PAPER SPECIFIC INSTRUCTIONS

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SECTION 'A'

1.(a) Prove that the set of all bijective functions from a non-empty set X onto itself is a group with respect to usual composition of functions. 8

1.(b) Examine the Uniform Convergence of

$$f_n(x) = \frac{\sin(nx+n)}{n}, \forall x \in \mathbb{R}, n = 1, 2, 3, \ldots$$ 8

1.(c) Find the maxima and minima of the function $f(x, y) = x^3 + y^3 - 3x - 12y + 20$. 8

1.(d) Find the analytic function of which the real part is

$$e^{-x}\left\{\left(x^2 - y^2\right)\cos y + 2xy\sin y\right\}.$$ 8

1.(e) Prove that the set of all feasible solutions of a Linear Programming problem is a convex set. 8

2.(a) Show that any non-abelian group of order 6 is isomorphic to the symmetric group S_3. 15

2.(b) Let G be a group of order pq, where p and q are prime numbers such that $p > q$ and $q \nmid (p-1)$. Then prove that G is cyclic. 15

2.(c) Show that in the ring $R = \left\{a + b\sqrt{-5} \mid a, b \text{ are integers}\right\}$, the elements $\alpha = 3$ and $\beta = 1 + 2\sqrt{-5}$ are relatively prime, but $\alpha\gamma$ and $\beta\gamma$ have no g.c.d in R, where $\gamma = 7\left(1 + 2\sqrt{-5}\right)$. 10

3.(a) If $f_n(x) = \dfrac{3}{x+n}$, $0 \leq x \leq 2$, state with reasons whether $\{f_n\}_n$ converges uniformly on [0, 2] or not. 10

3.(b) Examine the continuity of $f(x, y) = \begin{cases} \dfrac{\sin^{-1}(x+2y)}{\tan^{-1}(2x+4y)}, & (x, y) \neq (0, 0) \\ \frac{1}{2}, & (x, y) = (0, 0) \end{cases}$

at the point (0, 0). 8

3.(c) If $u(x, y) = \cos^{-1}\left\{\dfrac{x+y}{\sqrt{x}+\sqrt{y}}\right\}$, $0 < x < 1$, $0 < y < 1$ then find the value of

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}.$$ 10

3.(d) Evaluate the integral $\displaystyle\int_0^2\int_0^{y^2/2} \frac{y}{\left(x^2+y^2+1\right)^{\frac{1}{2}}}\,dx\,dy$. 12

4.(a) Evaluate the integral $\int_0^{\infty} \frac{dx}{\sqrt{x}(1+x)}$. 8

4.(b) Find the Laurent series for the function $f(z)=\frac{1}{1-z^2}$ with centre $z=1$. 10

4.(c) Evaluate by Contour integration $\int_0^{\pi} \frac{d\theta}{\left(1+\frac{1}{2}\cos\theta\right)^2}$. 10

4.(d) A company manufacturing air-coolers has two plants located at Bengaluru and Mumbai with a weekly capacity of 200 units and 100 units respectively. The company supplies air-coolers to its 4 showrooms situated at Mangalore, Bengaluru, Delhi and Goa which have a demand of 75, 100, 100 and 25 units respectively. Due to the differences in local taxes, showroom charges, transportation cost and others, the profits differ. The profits (in Rs.) are shown in the following table :

From	To			
	Mangalore	Bengaluru	Delhi	Goa
Bengaluru	90	90	100	100
Mumbai	50	70	130	85

Plan the production program so as to maximize the profit. The company may have its production capacity at both plants partially or wholly unused. 12

SECTION 'B'

5.(a) Obtain the partial differential equation governing the equations

$$\phi(u, v) = 0, \quad u = xyz,$$
$$v = x + y + z.$$ 8

5.(b) Find the general solution of the partial differential equation

$$xy^2 \frac{\partial z}{\partial x} + y^3 \frac{\partial z}{\partial y} = \left(zxy^2 - 4x^3\right).$$ 8

5.(c) Develop an algorithm for Newton-Raphson method to solve $\phi(x) = 0$ starting with initial iterate x_0, n be the number of iterations allowed, eps be the prescribed relative error and delta be the prescribed lower bound for $\phi'(x)$. 8

5.(d) Apply Lagrange's interpolation formula to find $f(5)$ and $f(6)$ given that $f(1) = 2$, $f(2) = 4$, $f(3) = 8$, $f(7) = 128$. 8

5.(e) Calculate the moment of inertia of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

(i) relative to the x-axis
(ii) relative to the y-axis and
(iii) relative to the origin 8

6.(a) Find the general solution of the partial differential equation

$$xy^2p + y^3q = (zxy^2 - 4x^3)$$

$$\left[p=\frac{\partial z}{\partial x},\ q=\frac{\partial z}{\partial y}\right]$$ 10

6.(b) Find the particular integral of $\frac{\partial^2 z}{\partial x^2}-2\frac{\partial^2 z}{\partial x \partial y}+\frac{\partial^2 z}{\partial y^2}=2x\cos y$. 10

6.(c) A uniform rod of length L whose surface is thermally insulated is initially at temperature $\theta = \theta_0$. At time $t=0$, one end is suddenly cooled to $\theta = 0$ and subsequently maintained at this temperature; the other end remains thermally insulated. Find the temperature distribution $\theta(x, t)$. 20

7.(a) Evaluate $\int_0^{0\cdot 6} \frac{dx}{\sqrt{1-x^2}}$ by Simpson's $\frac{1}{3}$rd rule, by taking 12 equal sub-intervals. 15

7.(b) Find the cube root of 10 up to 5 significant figures by Newton-Raphson method. 10

7.(c) Use the Classical Fourth-order Runge-Kutta method with $h=\cdot 2$ to calculate a solution at $x=\cdot 4$ for the initial value problem $\frac{dy}{dx}=x+y^2$ with initial condition $y=1$ when $x=0$. 15

8.(a) Find the moment of inertia of a right solid cone of mass M, height h and radius of whose base is a, about its axis. 14

8.(b) A bead slides on a wire in the shape of a cycloid described by the equations

$$x = a(\theta - \sin\theta)$$
$$y = a(1 + \cos\theta)$$

where $0 \leqslant \theta \leqslant 2\pi$ and the friction between the bead and the wire is negligible. Deduce Lagrange's equation of motion. 10

8.(c) A sphere is at rest in an infinite mass of homogeneous liquid of density ρ, the pressure at infinity being P. If the radius R of the sphere varies in such a way that $R = a + b\cos nt$, where $b < a$, then find the pressure at the surface of the sphere at any time. 16