I.F.S. EXAM-(M) 2018

FSI-P-MTH

MATHEMATICS

Paper – I

Time Allowed : ***Three*** *Hours*

Maximum Marks : ***200***

Question Paper Specific Instructions

Please read each of the following instructions carefully before attempting questions :

There are ***EIGHT*** *questions in all, out of which* ***FIVE*** *are to be attempted.*

Questions no. ***1*** *and* ***5*** *are* ***compulsory****. Out of the remaining* ***SIX*** *questions,* ***THREE*** *are to be attempted selecting at least* ***ONE*** *question from each of the two Sections A and B.*

Attempts of questions shall be counted in sequential order. Unless struck off, attempt of a question shall be counted even if attempted partly. Any page or portion of the page left blank in the Question-cum-Answer Booklet must be clearly struck off.

All questions carry equal marks. The number of marks carried by a question/part is indicated against it.

Answers must be written in ***ENGLISH*** *only.*

Unless otherwise mentioned, symbols and notations have their usual standard meanings.

Assume suitable data, if necessary, and indicate the same clearly.

SECTION A

Q1. (a) Show that the maximum rectangle inscribed in a circle is a square. *8*

(b) Given that Adj A = $\begin{bmatrix} 2 & 2 & 0 \\ 2 & 5 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ and det A = 2. Find the matrix A. *8*

(c) If f : [a, b] $\to$ R be continuous in [a, b] and derivable in (a, b), where $0 < a < b$, show that for $c \in (a, b)$

$$f(b) - f(a) = cf'(c) \log (b/a).$$ *8*

(d) Find the equations of the tangent planes to the ellipsoid

$$2x^2 + 6y^2 + 3z^2 = 27$$

which pass through the line

$$x - y - z = 0 = x - y + 2z - 9.$$ *8*

(e) Prove that the eigenvalues of a Hermitian matrix are all real. *8*

Q2. (a) Find the equation of the cylinder whose generators are parallel to the line $\frac{x}{1} = \frac{y}{-2} = \frac{z}{3}$ and whose guiding curve is $x^2 + y^2 = 4, z = 2$. *10*

(b) Show that the matrices

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 3 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 2 \\ 3 & 2 & 0 \end{bmatrix} \text{ are congruent.}$$ *10*

(c) If ϕ and ψ be two functions derivable in [a, b] and $\phi(x)\, \psi'(x) - \psi(x)\, \phi'(x) > 0$ for any x in this interval, then show that between two consecutive roots of $\phi(x) = 0$ in [a, b], there lies exactly one root of $\psi(x) = 0$. *10*

(d) Show that the vectors $\alpha_1 = (1, 0, -1)$, $\alpha_2 = (1, 2, 1)$, $\alpha_3 = (0, -3, 2)$ form a basis for R^3. Express each of the standard basis vectors as a linear combination of $\alpha_1, \alpha_2, \alpha_3$. *10*

Q3. (a) Find the equation of the tangent plane that can be drawn to the sphere

$$x^2 + y^2 + z^2 - 2x + 6y + 2z + 8 = 0,$$

through the straight line

$$3x - 4y - 8 = 0 = y - 3z + 2.$$

10

(b) If $f = f(u, v)$, where $u = e^x \cos y$ and $v = e^x \sin y$, show that

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = (u^2 + v^2)\left(\frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2}\right).$$

10

(c) Let $T : V_2(R) \to V_2(R)$ be a linear transformation defined by $T(a, b) = (a, a + b)$. Find the matrix of T, taking $\{e_1, e_2\}$ as a basis for the domain and $\{(1, 1), (1, -1)\}$ as a basis for the range. *10*

(d) Evaluate $\iint_R (x^2 + xy)\, dx\, dy$ over the region R bounded by $xy = 1$, $y = 0$, $y = x$ and $x = 2$. *10*

Q4. (a) Find the equations of the straight lines in which the plane $2x + y - z = 0$ cuts the cone $4x^2 - y^2 + 3z^2 = 0$. Find the angle between the two straight lines. *10*

(b) Show that the functions $u = x + y + z$, $v = xy + yz + zx$ and $w = x^3 + y^3 + z^3 - 3xyz$ are dependent and find the relation between them. *10*

(c) Find the locus of the point of intersection of the perpendicular generators of the hyperbolic paraboloid $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 2z$. *10*

(d) If $(n + 1)$ vectors $\alpha_1, \alpha_2, \ldots, \alpha_n, \alpha$ form a linearly dependent set, then show that the vector α is a linear combination of $\alpha_1, \alpha_2, \ldots, \alpha_n$; provided $\alpha_1, \alpha_2, \ldots, \alpha_n$ form a linearly independent set. *10*

SECTION B

Q5. (a) Find the complementary function and particular integral for the equation

$$\frac{d^2y}{dx^2} - y = xe^x + \cos^2 x$$

and hence the general solution of the equation. *8*

(b) Solve $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = xe^x \log x \ (x > 0)$ by the method of variation of parameters. *8*

(c) If the velocities in a simple harmonic motion at distances a, b and c from a fixed point on the straight line which is not the centre of force, are u, v and w respectively, show that the periodic time T is given by

$$\frac{4\pi^2}{T^2}(b - c)(c - a)(a - b) = \begin{vmatrix} u^2 & v^2 & w^2 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix}.$$ *8*

(d) From a semi-circle whose diameter is in the surface of a liquid, a circle is cut out, whose diameter is the vertical radius of the semi-circle. Find the depth of the centre of pressure of the remainder part. *8*

(e) If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and f(r) is differentiable, show that

$\text{div}[f(r)\vec{r}\,] = rf'(r) + 3f(r)$.

Hence or otherwise show that $\text{div}\left(\frac{\vec{r}}{r^3}\right) = 0$. *8*

Q6. (a) Solve the differential equation $(y^2 + 2x^2y)\,dx + (2x^3 - xy)\,dy = 0$. *10*

(b) Let T_1 and T_2 be the periods of vertical oscillations of two different weights suspended by an elastic string, and C_1 and C_2 are the statical extensions due to these weights and g is the acceleration due to gravity. Show that $g = \frac{4\pi^2(C_1 - C_2)}{T_1^2 - T_2^2}$. *15*

(c) Show that $\vec{F} = (2xy + z^3)\hat{i} + x^2\hat{j} + 3xz^2\hat{k}$ is a conservative force. Hence, find the scalar potential. Also find the work done in moving a particle of unit mass in the force field from (1, – 2, 1) to (3, 1, 4). *15*

Q7. (a) The end links of a uniform chain slide along a fixed rough horizontal rod. Prove that the ratio of the maximum span to the length of the chain is

$$\mu \log \frac{1 + (1 + \mu^2)^{\frac{1}{2}}}{\mu}$$

where μ is the coefficient of friction. *10*

(b) Solve : *10*

$$\frac{dy}{dx} = \frac{4x + 6y + 5}{3y + 2x + 4}$$

(c) A frame ABC consists of three light rods, of which AB, AC are each of length a, BC of length $\frac{3}{2}$ a, freely jointed together. It rests with BC horizontal, A below BC and the rods AB, AC over two smooth pegs E and F, in the same horizontal line, at a distance 2b apart. A weight W is suspended from A. Find the thrust in the rod BC. *10*

(d) Let α be a unit-speed curve in R^3 with constant curvature and zero torsion. Show that α is (part of) a circle. *10*

Q8. (a) A solid hemisphere floating in a liquid is completely immersed with a point of the rim joined to a fixed point by means of a string. Find the inclination of the base to the vertical and tension of the string. *15*

(b) A snowball of radius r(t) melts at a uniform rate. If half of the mass of the snowball melts in one hour, how much time will it take for the entire mass of the snowball to melt, correct to two decimal places ? Conditions remain unchanged for the entire process. *15*

(c) For a curve lying on a sphere of radius a and such that the torsion is never 0, show that

$$\left(\frac{1}{\kappa}\right)^2 + \left(\frac{\kappa'}{\kappa^2 \tau}\right)^2 = a^2.$$ *10*

I.F.S. EXAM-(M) 2018

FSI–D–MTH

MATHEMATICS

PAPER—II

Time Allowed : Three Hours	Maximum Marks : 200

QUESTION PAPER SPECIFIC INSTRUCTIONS

Please read each of the following instructions carefully before attempting questions

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Question Nos. **1** and **5** are compulsory. Out of the remaining **SIX** questions, **THREE** are to be attempted selecting at least **ONE** question from each of the two Sections **A** and **B**.

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SECTION—A

1. *(a)* Prove that a non-commutative group of order $2n$, where n is an odd prime, must have a subgroup of order n. 8

(b) A function $f : [0, 1] \rightarrow [0, 1]$ is continuous on $[0, 1]$. Prove that there exists a point c in $[0, 1]$ such that $f(c) = c$. 10

(c) If $u = (x-1)^3 - 3xy^2 + 3y^2$, determine v so that $u + iv$ is a regular function of $x + iy$. 10

(d) Solve by simplex method the following Linear Programming Problem : 12

Maximize $Z = 3x_1 + 2x_2 + 5x_3$

subject to the constraints

$$\begin{aligned} x_1 + 2x_2 + x_3 &\leq 430 \\ 3x_1 + 2x_3 &\leq 460 \\ x_1 + 4x_2 &\leq 420 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

2. *(a)* Find all the homomorphisms from the group $(\mathbb{Z}, +)$ to $(\mathbb{Z}_4, +)$. 10

(b) Consider the function f defined by

$$f(x, y) = \begin{cases} xy\dfrac{x^2 - y^2}{x^2 + y^2}, & \text{where } x^2 + y^2 \neq 0 \\ 0, & \text{where } x^2 + y^2 = 0 \end{cases}$$

Show that $f_{xy} \neq f_{yx}$ at $(0, 0)$. 10

(c) Prove that $\int_0^\infty \cos x^2 dx = \int_0^\infty \sin x^2 dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}$. 10

(d) Let R be a commutative ring with unity. Prove that an ideal P of R is prime if and only if the quotient ring R/P is an integral domain. 10

3. *(a)* Find the minimum value of $x^2 + y^2 + z^2$ subject to the condition $ax + by + cz = p$. 10

(b) Show by an example that in a finite commutative ring, every maximal ideal need not be prime. 10

(c) Evaluate the integral $\int_0^{2\pi} \cos^{2n} \theta \, d\theta$, where n is a positive integer. 10

(d) Show that the improper integral $\int_0^1 \frac{\sin \frac{1}{x}}{\sqrt{x}} dx$ is convergent. 10

4. *(a)* Show that

$$\iint_R x^{m-1} y^{n-1} (1-x-y)^{l-1} \, dx \, dy = \frac{\Gamma(l)\Gamma(m)\Gamma(n)}{\Gamma(l+m+n)}; \quad l, m, n > 0$$

taken over R : the triangle bounded by $x = 0$, $y = 0$, $x + y = 1$. 10

(b) Let $f_n(x) = \dfrac{x}{n + x^2}$, $x \in [0, 1]$. Show that the sequence $\{f_n\}$ is uniformly convergent on $[0, 1]$. 8

(c) Let H be a cyclic subgroup of a group G. If H be a normal subgroup of G, prove that every subgroup of H is a normal subgroup of G. 10

(d) The capacities of three production facilities S_1, S_2 and S_3 and the requirements of four destinations D_1, D_2, D_3 and D_4 and transportation costs in rupees are given in the following table :

	D_1	D_2	D_3	D_4	*Capacity*
S_1	19	30	50	10	7
S_2	70	30	40	60	9
S_3	40	8	70	20	18
Demand	5	8	7	14	34

Find the minimum transportation cost using Vogel's Approximation Method (VAM). 12

SECTION—B

5. *(a)* Find the partial differential equation of all planes which are at a constant distance a from the origin. 10

(b) A solid of revolution is formed by rotating about the x-axis, the area between the x-axis, the line $x = 0$ and a curve through the points with the following coordinates :

x	0·0	0·25	0·50	0·75	1·00	1·25	1·50
y	1·0	0·9896	0·9589	0·9089	0·8415	0·8029	0·7635

Estimate the volume of the solid formed using Weddle's rule. 10

(c) Write a program in BASIC to multiply two matrices (checking for consistency for multiplication is required). 10

(d) Air, obeying Boyle's law, is in motion in a uniform tube of small section. Prove that if ρ be the density and v be the velocity at a distance x from a fixed point at time t, then $\dfrac{\partial^2 \rho}{\partial t^2} = \dfrac{\partial^2}{\partial x^2} \{\rho(v^2 + k)\}$. 10

6. *(a)* Find the complete integral of the partial differential equation $(p^2 + q^2)x = zp$ and deduce the solution which passes through the curve $x = 0,\ z^2 = 4y$. Here $p = \frac{\partial z}{\partial x},\ q = \frac{\partial z}{\partial y}$. 12

(b) Apply fourth-order Runge-Kutta method to compute y at $x = 0 \cdot 1$ and $x = 0 \cdot 2$, given that $\frac{dy}{dx} = x + y^2,\ y = 1$ at $x = 0$. 12

(c) For a particle having charge q and moving in an electromagnetic field, the potential energy is $U = q(\phi - \vec{v} \cdot \vec{A})$, where ϕ and $\vec{A}$ are, respectively, known as the scalar and vector potentials. Derive expression for Hamiltonian for the particle in the electromagnetic field. 8

(d) Write a program in BASIC to implement trapezoidal rule to compute $\int_0^{10} e^{-x^2} dx$ with 10 subdivisions. 8

7. *(a)* Solve $(z^2 - 2yz - y^2)\, p + (xy + zx)\, q = xy - zx$, where $p = \frac{\partial z}{\partial x},\ q = \frac{\partial z}{\partial y}$.

If the solution of the above equation represents a sphere, what will be the coordinates of its centre? 8

(b) The velocity v (km/min) of a moped is given at fixed interval of time (min) as below :

t	0·1	0·2	0·3	0·4	0·5	0·6
v	1·00	1·104987	1·219779	1·34385	1·476122	1·615146

t	0·7	0·8	0·9	1·0	1·1
v	1·758819	1·904497	2·049009	2·18874	2·31977

Estimate the distance covered during the time (use Simpson's one-third rule). 10

(c) Assuming a 16-bit computer representation of signed integers, represent –44 in 2's complement representation. 10

(d) In the case of two-dimensional motion of a liquid streaming past a fixed circular disc, the velocity at infinity is u in a fixed direction, where u is a variable. Show that the maximum value of the velocity at any point of the fluid is $2u$. Prove that the force necessary to hold the disc is $2m\dot{u}$, where m is the mass of the liquid displaced by the disc. 12

8. *(a)* Find a real function V of x and y, satisfying $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = -4\pi(x^2 + y^2)$ and reducing to zero, when $y = 0$. 10

(b) The equation $x^6 - x^4 - x^3 - 1 = 0$ has one real root between 1·4 and 1·5. Find the root to four places of decimal by regula-falsi method. 10

(c) A particle of mass m is constrained to move on the inner surface of a cone of semi-angle α under the action of gravity. Write the equation of constraint and mention the generalized coordinates. Write down the equation of motion. 10

(d) Two sources, each of strength m, are placed at the points $(-a, 0)$, $(a, 0)$ and a sink of strength $2m$ at the origin. Show that the streamlines are the curves $(x^2 + y^2)^2 = a^2(x^2 - y^2 + \lambda xy)$, where λ is a variable parameter.

Show also that the fluid speed at any point is $(2ma^2)/(r_1 r_2 r_3)$, where r_1, r_2, r_3 are the distances of the point from the sources and the sink. 10

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